4.6 Variation of Parameters.

The method of undetermined coefficients has two weaknesses:

+ The coefficients must be constants
+ The function $g(x)$ must be of several special types.

In this section, we investigate a new method to find a particular solution of nonhomogeneous eqs, that has no such restriction on it. This method, due to Joseph Louis Lagrange (1736-1813), is known as **variation of parameters.**

LINEAR FIRST-ORDER DEs.

$$\frac{dy}{dx} + P(x)y = f(x) \qquad (1)$$

assuming that P and f are continuous on an interval I.
By integrating factor the general solution is

$$y = c_1 e^{-\int P(x)dx} + e^{-\int P(x)dx} \int e^{\int P(x)dx} f(x)dx$$

$\Big(\mu(x) = e^{\int P(x)dx}$;

$$\frac{d[\mu(x)y]}{dx} = \mu(x) f(x)$$

$$\Leftrightarrow \quad \mu(x)\, y = \int \mu(x) f(x)\, dx + c_1 \Big)$$

In this formula:

$$y = \underbrace{C_1 e^{-\int p(x)dx}}_{y_c} + \underbrace{e^{-\int p(x)dx} \int e^{\int p(x)dx} f(x)dx}_{y_p}$$

we have

$y_c = c_1 e^{-\int p(x)dx}$ is the general solution of the homogeneous eq corresponding to (1), i.e.

$$\frac{dy}{dx} + P(x)y = 0 , \qquad (2)$$

and $y_p = e^{-\int p(x)dx} \int e^{\int p(x)dx} f(x)dx \qquad (3)$

is a particular solution of (1). This is consistent to what we discuss in the beginnig of the chapter.

We now find the particular solution y_p by a new method known as **variation of parameter.**

⊕ Assume that y_1 is a known solution of the homogeneous eq (2).

It is easy to see that $y_1 = e^{-\int p(x)dx}$ is such a solution. And $c_1 y_1(x)$ is the general sol.

⊕ Variation of parameters consists of finding a particular solution of (1) of the form

$$y_p = u_1(x) y_1(x).$$

(i.e. we replace the parameter c_1 by a function u_1)

Substituting $y_p = u_1 y_1$ into (1), we get

$$\frac{d[u_1 y_1]}{dx} + P(x) u_1 y_1 = f(x)$$

$$\Leftrightarrow \quad u_1 \frac{dy_1}{dx} + y_1 \frac{du_1}{dx} + P u_1 y_1 = f(x)$$

$$\Leftrightarrow \quad u_1 \underbrace{\left[\frac{dy_1}{dx} + P y_1\right]}_{=0} + y_1 \frac{du_1}{dx} = f(x)$$

$$\Leftrightarrow \quad y_1 \frac{du_1}{dx} = f(x).$$

By separating variables, we find u_1:

$$du_1 = \frac{f(x)}{y_1} dx$$

$$\Leftrightarrow \quad u_1 = \int \frac{f(x)}{y_1} dx.$$

$$\Rightarrow \quad y_p = u_1 y_1 = y_1 \int \frac{f(x)}{y_1(x)} dx$$

that is identical to (3) as $y_1 = e^{-\int P(x)dx}$.

LINEAR SECOND-ORDER DEs

$$y'' + P(x) y' + Q(x) y = f(x) \qquad (4)$$

(We assume that P, Q, f are continuous on an interval I)

It is not hard to obtain the general solution

$$y_c = c_1 y_1 + c_2 y_2$$

of the associated homogeneous equation.
We would like to find a particular solution y_p of (4) in the form

$$y_p(x) = u_1(x) y_1(x) + u_2(x) y_2(x). \quad (5)$$

We have

$$y_p' = u_1 y_1' + u_1' y_1 + u_2 y_2' + u_2' y_2 \quad (6)$$

$$y_p'' = u_1 y_1'' + y_1' u_1' + y_1 u_1'' + u_1' y_1' + u_2 y_2'' + u_2' y_2' + y_2 u_2'' + u_2' y_2' \quad (7)$$

Substituting (5), (6), (7) into (4), we get

$$\begin{aligned}
y_p'' + P(x) y_p' + Q(x) y_p &= u_1 [y_1'' + P y_1' + Q y_1] + u_2 [y_2'' + P y_2' + Q y_2] \\
&\quad + y_1 u_1'' + u_1' y_1' + y_2 u_2'' + u_2' y_2' + P[y_1 u_1' + y_2 u_2'] \\
&\quad + y_1' u_1' + y_2' u_2' \\
&= \frac{d}{dx}[y_1 u_1'] + \frac{d}{dx}[y_2 u_2'] + P[y_1 u_1' + y_2 u_2'] \\
&\quad + y_1' u_1' + y_2' u_2' \\
&= \frac{d}{dx}[y_1 u_1' + y_2 u_2'] + P[y_1 u_1' + y_2 u_2'] + y_1' u_1' + y_2' u_2' \\
&= f(x). \quad (8)
\end{aligned}$$

We can make a further assumption that

$$y_1 u_1' + y_2 u_2' = 0.$$

Thus (8) is reduced to

$$y_1' u_1' + y_2' u_2' = f(x).$$

Thus we have a system with in u_1' and u_2':

$$y_1 u_1' + y_2 u_2' = 0$$
$$y_1' u_1' + y_2' u_2' = f(x).$$

Apply Cramer's Rule:

$$u_1' = \frac{W_1}{W} = -\frac{y_2 f(x)}{W} \qquad (9)$$
$$u_2' = \frac{W_2}{W_1} = \frac{y_1 f(x)}{W}$$

where
$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}, \quad W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}, \quad W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}$$
(10)

Example 1: Solve
$$y'' - 4y' + 4y = (x+1)e^{2x}$$

Solution:

From the auxiliary eq $m^2 - 4m + 4 = 0$, we have $y_c = C_1 e^{2x} + C_2 x e^{2x}$.

$$y_1 = e^{2x}, \quad y_2 = x e^{2x}$$

Next we comput the Wronskian

$$W(e^{2x}, xe^{2x}) = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & 2xe^{2x} + e^{2x} \end{vmatrix}$$
$$= e^{4x}.$$

$$W_1 = -(x+1)\,x\,e^{4x}$$
$$W_2 = (x+1)\,e^{4x}$$

By (9):

$$u_1' = -\frac{(x+1)x e^{4x}}{e^{4x}} = -x^2 - x$$

$$u_2' = \frac{(x+1) e^{4x}}{e^{4x}} = x+1$$

$$\Rightarrow \quad \left.\begin{aligned} u_1 &= -\frac{1}{3}x^3 - \frac{1}{2}x^2 \\ u_1 &= \frac{1}{2}x^2 + x \end{aligned}\right\}$$

we do not need to introduce constant parameters here!

$$\Rightarrow y_p = \left(-\frac{1}{3}x^3 - \frac{1}{2}x^2\right) e^{2x} + \left(\frac{1}{2}x^2 + x\right) x e^{2x}$$

$$= \frac{1}{6}x^3 e^{2x} + \frac{1}{2}x^2 e^{2x}.$$

and

$$y = y_c + y_p = c_1 e^{2x} + c_2 x e^{2x} + \frac{1}{6}x^3 e^{2x} + \frac{1}{2}x^2 e^{2x}.$$

$\square$

Example: Solve

$$4y'' + 36y = \csc 3x$$

Solution:

Put the equation in the standard form

$$y'' + 9y = \frac{1}{4}\csc 3x$$

The auxiliary eq

$$m^2 + 9m = 0$$

has two complex roots $m_1 = 3i$ and $m_2 = -3i$

the complementary function

$$y_c = c_1 \cos 3x + c_2 \sin 3x$$

we apply variation of parameters with $y_1 = \cos 3x$, $y_2 = \sin 3x$, and $f(x) = \frac{1}{4}\csc x$.

We obtain

$$W(\cos 3x, \sin 3x) = \begin{vmatrix} \cos 3x & \sin 3x \\ -3\sin 3x & 3\cos 3x \end{vmatrix} = 3$$

$$W_1 = \begin{vmatrix} 0 & \sin 3x \\ \frac{1}{4}\csc 3x & 3\cos 3x \end{vmatrix} = -\frac{1}{4}$$

$$W_2 = \begin{vmatrix} \cos 3x & 0 \\ -3\sin 3x & \frac{1}{4}\csc 3x \end{vmatrix} = \frac{1}{4} \frac{\cos 3x}{\sin 3x}$$

$$\Rightarrow \quad u_1' = \frac{W_1}{W} = -\frac{1}{12}, \quad u_2' = \frac{W_2}{W_1} = \frac{1}{12} \frac{\cos 3x}{\sin 3x}$$

Integrating u_1' and u_2' gives

$$u_1 = -\frac{1}{12}x \quad \text{and} \quad u_2 = \frac{1}{36}\ln|\sin 3x|$$

Thus the particular solution is

$$y_p = u_1 y_1 + u_2 y_2 = -\frac{1}{2}x\cos 3x + \frac{1}{36}(\sin 3x)\ln|\sin 3x|$$

The general solution of the eq is

$$y = y_c + y_p = c_1\cos 3x + c_2\sin 3x - \frac{1}{12}x\cos 3x + \frac{1}{36}(\sin 3x)\ln|\sin 3x|. \quad \square$$

<u>Note that:</u> The method of undetermined coefficients does NOT work for the function $g(x) = \csc x$.

Remark: When computing the indefinite integrals of u_1' and u_2', we do not need to introduce any constants. This is because

$$y = y_c + y_p = c_1 y_1 + c_2 y_2 + (u_1 + a_1) y_1 + (u_2 + a_2) y_2$$
$$= (c_1 + a_1) y_1 + (c_2 + a_2) y_2 + u_1 y_1 + u_2 y_2$$
$$= C_1 y_1 + C_2 y_2 + u_1 y_1 + u_2 y_2 .$$

INTEGRAL-DEFINED FUNCTIONS

We can write u_1 and u_2 as integral-defined functions

$$u_1 = -\int_{x_0}^{x} \frac{y_2(t) f(t)}{W(t)}\, dt \quad \text{and}$$

$$u_2 = \int_{x_0}^{x} \frac{y_1(t) f(t)}{W(t)}\, dt .$$

Here we assume that the integrand is continuous on the interval $[x_0, x]$.

Example. Solve $y'' - y = \frac{1}{x}$.

Solution: The auxiliary eq: $m^2 - 1 = 0$ has two roots $m_1 = 1$, $m_2 = -1$. Thus the complementary function $y_c = c_1 e^x + c_2 e^{-x}$. Now $W(e^x, e^{-x}) = -2$ and

$$u_1 = \frac{1}{2}\int_{x_0}^{x} \frac{e^{-t}}{t}\, dt , \quad u_2 = -\frac{1}{2}\int_{x_0}^{x} \frac{e^{t}}{t}\, dt$$

Since the integrals here are not elementary, we are forced to write

$$y_p = \frac{1}{2} e^{x} \int_{x_0}^{x} \frac{e^t}{t}\,dt - \frac{1}{2} e^{-x} \int_{x_0}^{x} \frac{e^t}{t}\,dt$$

and so

$$y = y_c + y_p = c_1 e^{x} + c_2 e^{-x} + \frac{1}{2} e^{x} \int_{x_0}^{x} \frac{e^t}{t}\,dt - \frac{1}{2} e^{-x} \int_{x_0}^{x} \frac{e^t}{t}\,dt.$$

HIGHER-ORDER EQUATIONS

$$y^{(n)} + P_{n-1}(x)\, y^{n-1} + \cdots + P_1(x)\, y' + P_0(x)\, y = f(x)$$

If $y_c = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$ is the complementary function, then a particular solution is

$$y_p(x) = u_1(x)\, y_1(x) + \cdots + u_2\, y_n(x),$$

where u_k' $(k = 1, \ldots, n)$ are determined by the system:

$$\begin{aligned}
y_1 u_1' + y_2 u_2' + \cdots + y_n u_n' &= 0 \\
y_1' u_1' + y_2' u_2' + \cdots + y_n' u_n' &= 0 \\
\vdots \qquad\qquad \vdots \qquad\qquad & \vdots \\
y_1^{(n-1)} u_1' + y_2^{(n-1)} u_2' + \cdots + y_n^{(n-1)} u_n' &= f(x)
\end{aligned}$$

The general form of Cramer's Rule gives

$$u_k' = \frac{W_k}{W}, \quad k = 1, 2, \ldots, n$$

$$W = W(y_1, \ldots, y_n)$$

W_k is obtained by replacing the k-th column in W by the column consisting of $(0, 0, \ldots, 0, f(x))$.

In particular, for $n=3$, we have

$$u_1' = \frac{W_1}{W}, \quad u_2' = \frac{W_2}{W}, \quad u_3' = \frac{W_3}{W},$$

where

$$W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}, \quad W_1 = \begin{vmatrix} 0 & y_2 & y_3 \\ 0 & y_2' & y_3' \\ f(x) & y_2'' & y_3'' \end{vmatrix} \begin{matrix} 0 & y_2 \\ 0 & y_2' \\ f(x) & y_2'' \end{matrix}$$

$$= y_2 y_3' f(x) - y_2' y_3 f(x)$$

$$W_2 = \begin{vmatrix} y_1 & 0 & y_3 \\ y_1' & 0 & y_3' \\ y_1'' & f(x) & y_3'' \end{vmatrix} \begin{matrix} y_1 & 0 \\ y_1' & 0 \\ y_1'' & f(x) \end{matrix} = y_1' y_3 f(x) - y_1 y_3' f(x)$$

$$W_3 = \begin{vmatrix} y_1 & y_2 & 0 \\ y_1' & y_2' & 0 \\ y_1'' & y_2'' & f(x) \end{vmatrix} \begin{matrix} y_1 & y_2 \\ y_1' & y_2' \\ y_1'' & y_2'' \end{matrix} = y_1 y_2' f(x) - y_1' y_2 f(x)$$